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# Complexified Penner's coordinates and its applications (Analysis and Topology of Discrete Groups and Hyperbolic Spaces)

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# Complexified Penner's coordinates and its applications

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## 1 Penner's $\lambda$ -lengths

### 1.1 A coordinate-system for Teichmüller space

Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the unit disk, a model of hyperbolic plane and

$$SU(1, 1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}.$$

Then  $PSU(1, 1)$  is the group of orientation preserving hyperbolic motions of  $\mathbb{D}$ .

Let  $G = G_{g,n}$  be the punctured surface group of type  $(g, n)$ , where  $2g - 2 + n > 0$ :

$$G = \langle a_1, b_1, \dots, a_g, b_g, d_1, \dots, d_n : \left( \prod_{k=1}^g a_k b_k a_k^{-1} b_k^{-1} \right) d_1 \cdots d_n = 1 \rangle.$$

A point of the *Teichmüller space*  $\mathcal{T} = \mathcal{T}_{g,n}$  is a class of faithful Fuchsian representations of  $G$  into  $PSU(1, 1)$  which have finite covolume. We denote points in  $\mathcal{T}$  by *marked groups*  $\Gamma_m$ , where  $\Gamma$  is a Fuchsian group and  $m : G \rightarrow \Gamma$  is an isomorphism.

Elements  $D_1, \dots, D_n$  in  $\Gamma_m \in \mathcal{T}$  corresponding to  $d_1, \dots, d_n$  are *parabolic*. Choose a horocycle  $H_k$  invariant under  $D_k$  such that action of  $D_k$  on  $H_k$  is the translation of length one. Then the identification of  $\Gamma_m$  with  $(\Gamma_m, H_1, \dots, H_n)$  gives the following statement.

$\mathcal{T}_{g,n}$  is naturally embedded in the decorated Teichmüller space  $\tilde{\mathcal{T}}_{g,n}$ .

Therefore, by restricting them to this embedded subspace, Penner's  $\lambda$ -length coordinates for  $\tilde{\mathcal{T}}_{g,n}$  give also global coordinates for the Teichmüller space  $\mathcal{T}_{g,n}$ .

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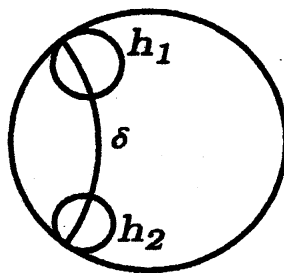
## 1.2 Distance between horocycles

Let  $p$  be a point of the unit circle. A *horocycle*  $h$  at  $p$  is a Euclidean circle in  $\mathbb{D}$  tangent at  $p$  to the unit circle. The point  $p$  is called the *base point* of  $h$ .

Let  $h_1$  and  $h_2$  be horocycles based at different points  $p_1$  and  $p_2$  and  $\gamma$  the hyperbolic line between  $p_1$  and  $p_2$ . Define

$$\lambda = e^{\delta/2}, \quad (1)$$

where  $\delta$  is the *signed* length of the portion of the geodesic  $\gamma$  intercepted between the two horocycles  $h_1$  and  $h_2$ ,  $\delta > 0$  if  $h_1$  and  $h_2$  are disjoint and  $\delta < 0$  otherwise. In this way we can assign a positive number  $\lambda$  to the pair  $(h_1, h_2)$ .



## 1.3 $\lambda$ -length of an ideal arc

Let  $S$  be the oriented closed surface of genus  $g$ ,  $P = \{p_1, \dots, p_n\}$  a set of  $n$  points. An *ideal arc*  $c$  of  $(S, P)$  is a path joining two points  $p_i$  and  $p_j$  in  $S - P$ . The ideal arc  $c$  is *simple* if  $c \cap (S - P)$  is a simple arc.

Let  $\Gamma_m \in \mathcal{T}_{g,n}$ , then there exists an orientation preserving homeomorphism

$$f : S - P \rightarrow \mathbb{D}/\Gamma$$

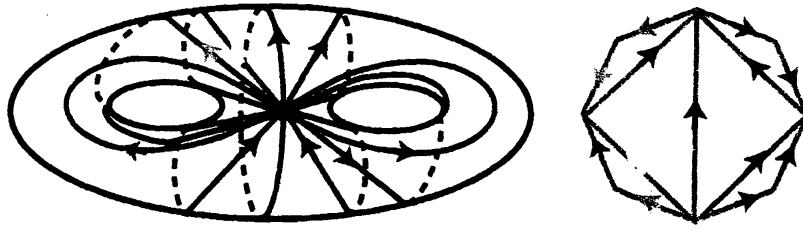
inducing  $m$ . Let  $\gamma$  be the geodesic representative in the homotopy class of  $f(c)$  for the Poincaré metric of the punctured surface  $\mathbb{D}/\Gamma$ . By the identification of  $\Gamma_m$  with  $(\Gamma_m, H_1, \dots, H_n)$ , the horocycles at the endpoints of  $\gamma$  defines the  $\lambda$ -length  $\lambda(c, \Gamma_m)$ .

Let  $\Delta = \{c_1, c_2, \dots, c_q\}$ ,  $q = 6g - 6 + 3n$ , be an ideal triangulation of  $(S, P)$ . Then

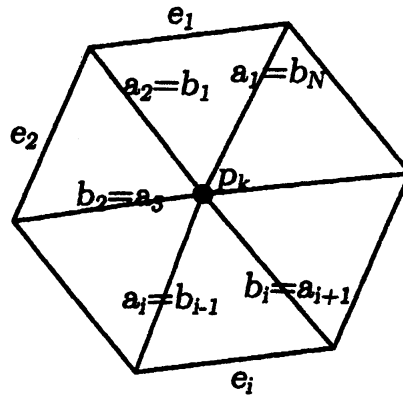
**Theorem 1** (Penner [1])

$$\lambda_\Delta = \prod_{i=1}^q \lambda(c_i) : \mathcal{T}_{g,n} \rightarrow (\mathbb{R}_+)^q$$

is an embedding.



The image of  $\lambda_\Delta$  is a real algebraic variety determined by  $n$  polynomials. A component of  $S - \cup_{j=1}^q c_j$  is called a *triangle* in  $\Delta$ . The image of  $\lambda_\Delta$  is a real algebraic variety determined by zero loci of  $n$  algebraic equations  $D_1, \dots, D_n$ , where  $D_k$  is easily obtained by triangles abutting on the  $k$ th puncture  $p_k$ .



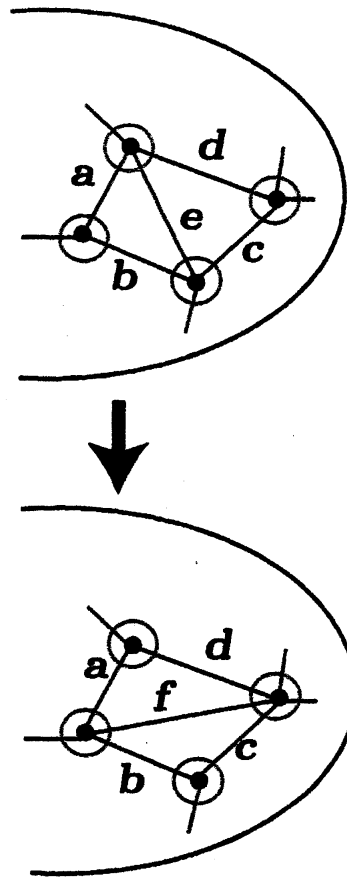
$$D_k(\lambda_1, \dots, \lambda_q) = \sum_{i=1}^N \frac{\lambda(e_i)}{\lambda(a_i)\lambda(b_i)} - 1. \quad (2)$$

#### 1.4 The Ptolemy identity

Let  $\Delta = \{c_1, c_2, \dots, c_q\}$  be an ideal triangulation of  $(S, P)$ . Let  $e \in \Delta$  and  $T_1$  and  $T_2$  be triangles being on the different sides of  $e$ . It is possible that  $T_1 = T_2$ . Lift  $T_1 \cup e \cup T_2$  to a quadrangle  $Q = \tilde{T}_1 \cup \tilde{e} \cup \tilde{T}_2$  in  $\mathbb{D}$ . Then  $\tilde{e}$  is a diagonal of  $Q$ . Let  $\tilde{f}$  be the other diagonal and project  $\tilde{f}$  to an ideal arc  $f$  in  $T_1 \cup e \cup T_2$ . Then

$$\Delta' = (\Delta - \{e\}) \cup \{f\}$$

is another ideal triangulation of  $(S, P)$ . We say that  $\Delta'$  arises from  $\Delta$  by the *elementary move* on  $e$



Let  $(\tilde{a}, \tilde{b}, \tilde{e})$  be the sides of  $\tilde{T}_1$  and  $(\tilde{c}, \tilde{d}, \tilde{e})$  be the sides of  $\tilde{T}_2$ . Suppose that  $\tilde{a}$  and  $\tilde{c}$  are opposite sides of  $Q$ . Let  $a, b, c, d \in \Delta \cap \Delta'$  be the projections of  $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ . The following theorems are proved in Penner's paper :

**Theorem 2** (the Ptolemy identity, Penner [1])  
*The  $\lambda$ -lengths function satisfy the identity*

$$\lambda(a)\lambda(c) + \lambda(b)\lambda(d) = \lambda(e)\lambda(f) \quad (3)$$

This theorem describes the coordinate-change between  $\lambda_\Delta(T)$  and  $\lambda_{\Delta'}(T)$ :

$$\begin{aligned} & \lambda_{\Delta'} \circ \lambda_\Delta^{-1}(\dots, \lambda(a), \lambda(b), \lambda(c), \lambda(d), \lambda(e), \dots) \\ &= (\dots, \lambda(a), \lambda(b), \lambda(c), \lambda(d), \frac{\lambda(a)\lambda(c) + \lambda(b)\lambda(d)}{\lambda(e)}, \dots) \end{aligned} \quad (4)$$

**Theorem 3** (Penner [1]) *For arbitrary ideal triangulations  $\Delta$  and  $\Delta'$  of  $(S, P)$ , there exists a finite sequence of ideal triangulations*

$$\Delta = \Delta_0, \Delta_1, \dots, \Delta_m = \Delta',$$

*where each  $\Delta_i$  arises from  $\Delta_{i-1}$  by an elementary move.*

Using this theorem it can be shown that coordinate change between  $\lambda$ -length coordinates associated with two ideal triangulations is a bi-rational map:

**Theorem 4** *If  $\Delta$  and  $\Delta'$  are ideal triangulations of  $F$ , then the coordinate change*

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\lambda_\Delta} & \lambda_\Delta(\mathcal{T}) \subset (\mathbb{R}_+)^q \\ \text{id} \downarrow & & \downarrow \lambda_{\Delta'} \circ \lambda_\Delta^{-1} \\ \mathcal{T} & \xrightarrow{\lambda_{\Delta'}} & \lambda_{\Delta'}(\mathcal{T}) \subset (\mathbb{R}_+)^q \end{array}$$

*extends to a rational transformation of  $\mathbb{R}^q$*

Let  $\mathcal{MC} = \mathcal{MC}_{g,n}$  denote the *mapping class group* of  $(S, P)$ . Each  $\varphi \in \mathcal{MC}$  acts on the Teichmüller space  $\mathcal{T}$ . The theorem above yields

**Theorem 5** *The correspondence*

$$\phi \mapsto \phi_* = \lambda_{\varphi^{-1}(\Delta)} \circ \lambda_\Delta^{-1}$$

*gives an isomorphism of  $\mathcal{MC}$  to a group of rational transformations.*

## 2 $SL(2, \mathbb{C})$ -representation space of a punctured surface group

Let  $\mathcal{R} = \mathcal{R}_{g,n}$  be the space of classes of faithful representations  $[m]$  of the punctured surface group  $G$  into  $SL(2, \mathbb{C})$  such that  $m(d_i)$  is parabolic with  $\text{tr } m(d_i) = -2$  for  $i = 1, 2, \dots, n$ . The Teichmüller space  $\mathcal{T}_{g,n}$  is a subspace of  $\mathcal{R}_{g,n}$ .

Our purpose is to give a coordinate-system for  $\mathcal{R}_{g,n}$  whose restriction to  $\mathcal{T}_{g,n}$  coincides with Penner's  $\lambda$ -lengths coordinate-system.

### 2.1 Parabolic elements of $SL(2, \mathbb{C})$

Define

$$\mathcal{P} = \{P \in SL(2, \mathbb{C}) : P \text{ is parabolic with } \text{tr } P = -2\}.$$

If  $P_1$  and  $P_2 \in \mathcal{P}$  do not commute, then the square root of  $-P_1 P_2$  in  $SL(2, \mathbb{C})$

$$Q = \pm \frac{1}{\sqrt{2 - \text{tr } P_1 P_2}} (I - P_1 P_2), \quad (5)$$

is unique up to sign and satisfies

$$P_2 = Q^{-1} P_1 Q. \quad (6)$$

For the rest of this paper, the diagram

$$P_1 \xrightarrow{Q} P_2$$

will mean that  $Q^2 = -P_1P_2$ .

### Cycles of parabolic elements

Let  $P_1, \dots, P_n, P_{n+1} = P_1 \in \mathcal{P}$ . Suppose that no consecutive elements  $P_i$  and  $P_{i+1}$  commute. Let  $Q_i$  be a square root of  $-P_iP_{i+1}$ , ( $i = 1, 2, \dots, n$ ). Then, since  $P_{i+1} = Q_i^{-1}P_iQ_i$ ,  $Q_1Q_2 \cdots Q_n$  commutes with  $P_1$ ,

$$\text{tr} Q_1Q_2 \cdots Q_n = +2 \text{ or } -2. \quad (7)$$

### Definition

$(Q_1, Q_2, \dots, Q_n)$  is a  $(+)$ -system or a  $(-)$ -system according to if  $\text{tr} Q_1Q_2 \cdots Q_n = +2$  or  $-2$ .

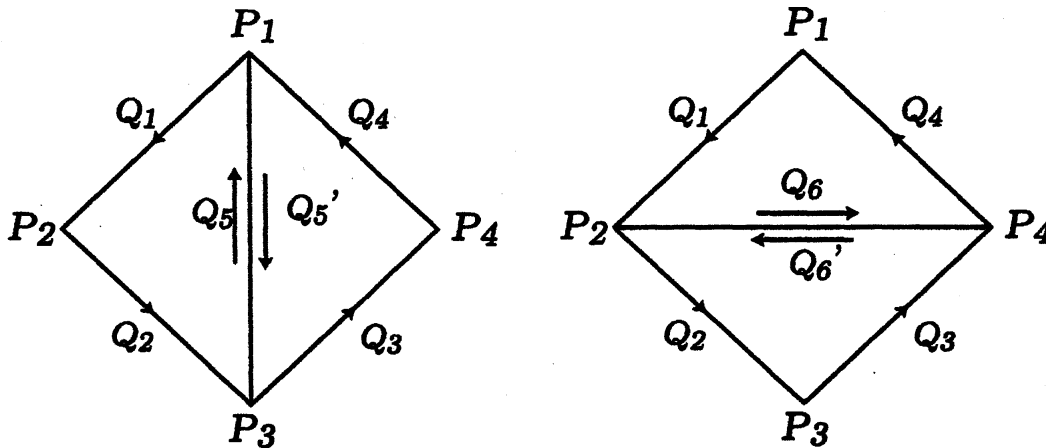
## 2.2 A trace identity of Ptolemy type

Let  $P_1, P_2, P_3$  and  $P_4$ . Suppose that  $P_i$  and  $P_j$  do not commute unless  $i = j$ . Choose  $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q'_5, Q'_6 \in SL(2, \mathbb{C})$  so that

$$\begin{aligned} Q_1^2 &= -P_1P_2, & Q_2^2 &= -P_2P_3, & Q_3 &= -P_3P_4, \\ Q_4^2 &= -P_4P_1, & Q_5^2 &= -P_3P_1, & Q_6 &= -P_2P_4, \\ (Q'_5)^2 &= -P_1P_3, & (Q'_6)^2 &= -P_4P_2, \end{aligned}$$

where

$$Q'_5 = P_1Q_5P_1^{-1}, \quad Q'_6 = P_4Q_6P_4^{-1}.$$



**Theorem 6** If  $(Q_1, Q_2, Q_5)$ ,  $(Q'_5, Q_3, Q_4)$  and  $(Q_1, Q_6, Q_4)$  are  $(-)$ -systems, then

$$\text{tr} Q_5 \text{tr} Q_6 = \text{tr} Q_1 \text{tr} Q_3 + \text{tr} Q_2 \text{tr} Q_4 \quad (8)$$

### 3 Complexified $\lambda$ -length

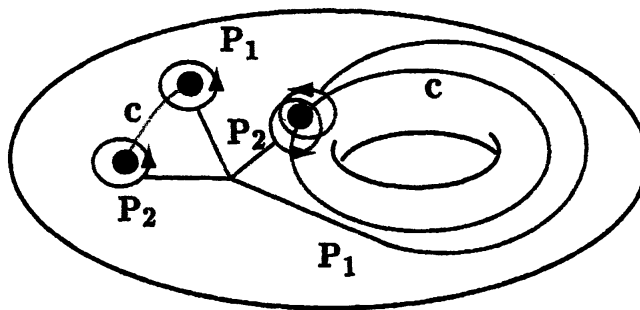
#### 3.1 Definition of $\lambda$ -length

A point of  $\mathcal{R}$  is represented by a marked group  $\Gamma_m$ . Let  $\mathcal{P}_+(\Gamma)$  be the set of parabolic elements in  $[m(d_1)] \cup \dots \cup [m(d_n)]$ , where  $[m(d_i)]$  is the conjugacy class of  $m(d_i)$ .

Let  $c$  be an ideal arc in  $(S, P)$ . Then for each  $\Gamma_m \in \mathcal{R}$ ,  $c$  defines two parabolic elements  $P_1, P_2$  of  $\mathcal{P}_+(\Gamma)$ , see the following figure. We define the  $\lambda$ -length of  $c$  with respect to  $\Gamma_m$  by

$$\lambda(c, \Gamma_m) = \text{tr} Q, \quad (9)$$

where  $Q$  is a square root of  $-P_1 P_2$ . The  $\lambda$ -length is defined up to sign.



#### 3.2 $\lambda$ -length coordinates for $\mathcal{R}_{g,n}$

Let  $\Delta = (c_1, c_2, \dots, c_q)$  be an ideal triangulation of  $(S, P)$ . Let  $T$  be a triangle in  $\Delta$ .  $T$  inherits the orientation of the surface  $S$ . Label the sides of  $T$  by  $a, b, c$  in order. Then those sides determine matrices  $Q_a, Q_b, Q_c$  whose traces give  $\lambda$ -lengths of  $a, b$  and  $c$  for  $\Gamma_m$ .

**Lemma 1** *It is possible to choose branches of  $\lambda$ -length functions  $\lambda(c_1), \lambda(c_2), \dots, \lambda(c_q)$  so that  $(Q_a, Q_b, Q_c)$  is a  $(-)$ -system for each triangle  $T$  in  $\Delta$ .*

With the choice of branches of  $\lambda$ -lengths as depicted in the lemma, we obtain

**Theorem 7** *For each ideal triangulation  $\Delta$ ,*

$$\lambda_\Delta = \prod_{i=1}^q \lambda(c_i) : \mathcal{R}_{g,n} \rightarrow (\mathbb{C}^*)^q$$

*is an embedding. The image is contained in an algebraic variety.*

#### 3.3 Rational representation of the mapping class group

As in the case of  $\mathcal{T}$ , the Ptolemy identity (8) yields

**Theorem 8** *The mapping class group  $\mathcal{MC}$  acts on  $\mathcal{R}$  as a group of rational transformations.*



## 4 Invariant holomorphic two-form

Let  $T_1, \dots, T_p$ ,  $p = 4g - 2$ , be triangles in an ideal triangulation of a once-punctured surface. Let the sequence of sides  $a_i, b_i, c_i$  of  $T_i$  agree with the positive orientation of  $T_i$ , then the 2-form

$$\sum_{i=1}^p (d \log \lambda(a_i) \wedge d \log \lambda(b_i) + d \log \lambda(b_i) \wedge d \log \lambda(c_i) + d \log \lambda(c_i) \wedge d \log \lambda(a_i)) \quad (10)$$

is invariant under the mapping class group  $\mathcal{MC}$ . The proof is similar to the one of the corresponding result in [2].

## 5 A characterization of the rational map induced by a mapping class

### 5.1 Example: Once punctured torus

The Teichmüller space  $\mathcal{T}_{1,1}$  of once punctured tori is represented as the subspace of  $(\mathbb{R}_+)^3$  defined by

$$x^2 + y^2 + z^2 = xyz, \quad (11)$$

where  $x, y, z$  are  $\lambda$ -length functions related to an (essentially unique) triangulation of the once punctured torus (or  $x, y, z$  are trace functions  $\text{tr}_A, \text{tr}_B, \text{tr}_{AB}$ , with  $\{A, B\}$  the canonical generator-system of  $G_{1,1}$ .)

The mapping class group  $\mathcal{MC}_{1,1}$  has generators

$$\sigma(x, y, z) = (x, z, \frac{x^2 + z^2}{y}) \quad \text{and} \quad \tau(x, y, z) = (\frac{x^2 + y^2}{z}, y, x),$$

with relations

$$(\tau \circ \sigma)^3 = 1, \quad (\sigma \circ \tau \circ \sigma)^2 = 1.$$

Since  $\mathcal{MC}_{1,1}$  acts on  $\mathcal{T}_{1,1}$ , the group of rational transformations generated by  $\sigma$  and  $\tau$  preserves the equation (11) and  $(x, y, z) = (3, 3, 3)$  gives integer solutions of (11).

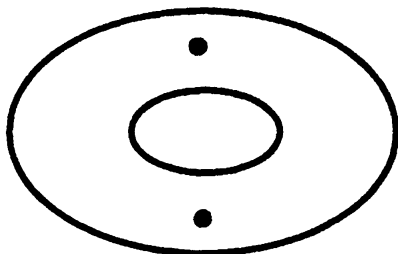
**Theorem 9** (Markoff) *All positive integer solutions of (11) are in the orbit of  $(3, 3, 3)$  under the action of  $\mathcal{MC}_{1,1}$ .*

The viewpoint of understanding the Markoff transformations as mapping classes acting on  $\mathcal{T}_{1,1}$  is given in Penner's paper [1].

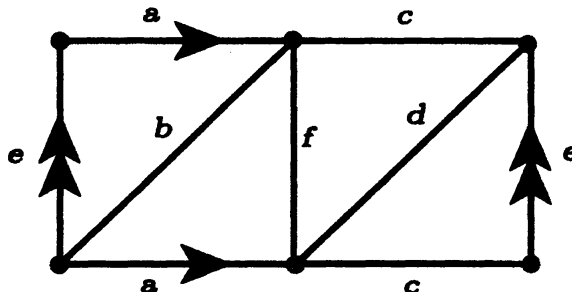
With  $\lambda$ -length coordinates, the Teichmüller space  $\mathcal{T}_{g,n}$  is determined by  $n$  algebraic equations and the group of rational transformations induced by the mapping class group  $\mathcal{MC}_{g,n}$  keep this space. So we can pursue analogies of the above result.

## 5.2 Example: twice punctured torus

Let  $\Delta$  be the ideal triangulation of the twice punctured torus as depicted in the following figure.



*twice punctured torus*



Consider the  $\lambda$ -lengths

$$\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e$$

associated with  $\Delta$ . Then it holds that  $\lambda_e = \lambda_f$ . The Teichmüller space  $\mathcal{T}_{1,2}$  (or the space  $\mathcal{R}_{1,2}$ ) is represented by the  $\lambda$ -lengths as the space

$$\frac{\lambda_e}{\lambda_a \lambda_b} + \frac{\lambda_a}{\lambda_b \lambda_e} + \frac{\lambda_b}{\lambda_a \lambda_e} + \frac{\lambda_c}{\lambda_d \lambda_e} + \frac{\lambda_d}{\lambda_c \lambda_e} + \frac{\lambda_e}{\lambda_c \lambda_d} = 1$$

or

$$\lambda_c \lambda_d (\lambda_a^2 + \lambda_b^2 + \lambda_e^2) + \lambda_a \lambda_b (\lambda_c^2 + \lambda_d^2 + \lambda_e^2) = \lambda_a \lambda_b \lambda_c \lambda_d \lambda_e. \quad (12)$$

The mapping class group  $\mathcal{MC}_{1,2}$  (as a group of rational transformations) has generators

$$\begin{aligned} \omega_{1*}(\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e) &= (\lambda_d, \lambda_b, \lambda_c, \frac{\lambda_e^2 + \lambda_d^2}{\lambda_a}, \lambda_e) \\ \omega_{2*}(\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e) &= (\lambda_d, \lambda_a, \lambda_b, \lambda_c, \frac{\lambda_a \lambda_c + \lambda_b \lambda_e}{\lambda_e}) \\ \omega_{3*}(\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e) &= (\lambda_a, \frac{\lambda_b^2 + \lambda_e^2}{\lambda_c}, \lambda_b, \lambda_d, \lambda_e), \end{aligned}$$

with relations

$$\omega_{2*}^2 \omega_{1*} \omega_{2*}^2 = \omega_{3*} \quad \omega_{1*} \omega_{3*} = \omega_{3*} \omega_{1*}$$

$$(\omega_{1*} \omega_{2*})^3 = 1, \quad (\omega_{3*} \omega_{2*})^3 = 1$$

The point  $p = (6, 6, 6, 6, 6)$  gives integer solutions of (12). An analogous result to the Markoff equation holds:

**Theorem 10** *The orbit  $\{\varphi_*(6, 6, 6, 6, 6) : \varphi \in \mathcal{MC}_{1,2}\}$ , gives integer solutions of (12), but not all of its integer solutions.*

### 5.3 Diophantine equations

We consider a once punctured surface.

**Lemma 2** *Let  $(\lambda_1, \lambda_2, \dots, \lambda_q)$  be the  $\lambda$ -length coordinate-system for  $\mathcal{R}_{g,1}$  associated to an ideal triangulation  $(c_1, c_2, \dots, c_q)$ , where  $q = 6g - 3$ . Then the  $\lambda$ -length of a simple ideal arc  $c$  is expressed by a rational function of the form*

$$\frac{P(\lambda_1, \lambda_2, \dots, \lambda_q)}{\lambda_1^{m_1} \lambda_2^{m_2} \dots \lambda_q^{m_q}}, \quad (13)$$

where  $P(\lambda_1, \lambda_2, \dots, \lambda_q)$  is a homogeneous polynomial of degree

$$d = 1 + m_1 + m_2 + \dots + m_q,$$

with positive integer coefficients and  $m_i$  is the geometric intersection number of  $c$  and  $c_i$  in  $S - P$  for  $i = 1, 2, \dots, q$

For  $\varphi \in \mathcal{MC}_{g,1}$  let  $\varphi_*$  denote the rational transformation induced by  $\varphi$ . Then entries of  $\varphi_*(\lambda_1, \lambda_2, \dots, \lambda_q)$  are of the form as in (13). This fact leads us to the following observation.

Let

$$D(\lambda_1, \dots, \lambda_q) = 0 \quad (14)$$

be the algebraic equation which determines  $\mathcal{T}_{g,1}$  in the  $\lambda$ -length coordinates. Then the rational transformation  $\varphi_*$  induced by  $\varphi \in \mathcal{MC}_{g,1}$  preserves  $D(\lambda_1, \dots, \lambda_q)$ . Moreover, if

$$(\lambda, \lambda, \dots, \lambda)$$

gives integer solutions of (14), then so does  $\varphi_*(\lambda, \lambda, \dots, \lambda)$ .

We remark that it is not true in general that all integer solutions are in the orbit of  $(\lambda, \lambda, \dots, \lambda)$  under  $\mathcal{MC}$ .

## 6 3-manifolds which fiber over the circle

Let  $\varphi \in \mathcal{MC}_{g,n}$ . Let  $M_\varphi$  be a manifold which fibers over the circle and whose monodromy is  $\varphi$ . If  $\varphi_*$  denotes the action of  $\varphi$  on the fundamental group  $G = G_{g,n}$  of the surface  $S$  of type  $(g, n)$ , then the fundamental group of  $M_\varphi$  has the presentation

$$\tilde{G} = \langle G, t : \varphi_*(g) = tgt^{-1} \text{ for all } g \in G \rangle \quad (15)$$

If  $m : \tilde{G} \rightarrow SL(2, \mathbb{C})$  is a faithful representation of  $\tilde{G}$ , then for all  $g \in G$

$$(\varphi_* \circ m)(g) = m(t)m(g)m(t)^{-1}.$$

Hence the class  $[m]$  is a fixed point of  $\varphi_*$  for its action on  $\mathcal{R}_{g,n}$ .

The  $\lambda$ -length coordinates of  $\mathcal{R}_{g,n}$  represent  $\varphi_*$  as a rational function. Hence the fixed point  $[m]$  corresponds to a solution of the algebraic equation

$$\varphi_*(\lambda_1, \dots, \lambda_q) = (\lambda_1, \dots, \lambda_q). \quad (16)$$

If  $\varphi$  is reducible, then one of the solutions of (16) gives a faithful and *discrete* representation  $m$  of  $G$ . We can find the Möbius transformation  $m(t)$  easily, because  $m(t)$  sends the fixed point of  $m(g)$  to that of  $m(\varphi_*(g))$  for each parabolic element  $g \in G$ . In this way *hyperbolization* of  $M_\varphi$  can be done. However, to carry this hyperbolization program into effect, we need efficient discreteness criteria.

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